Valuation bases for generalized algebraic series fields*

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Abstract

We investigate valued fields which admit a valuation basis. Given a countable ordered abelian group G and a real closed, or algebraically closed field F, we give a sufficient condition for a valued subfield of the field of generalized power series $F((G))$ to admit a K-valuation basis. We show that the field of rational functions $F(G)$ and the field $F(G)^\sim$ of power series in $F((G))$ algebraic over $F(G)$ satisfy this condition. It follows that for archimedean F and divisible G the real closed field $F(G)^\sim$ admits a restricted exponential function.

1 Introduction

Before describing the motivation for this research, and stating the main results obtained, we need to briefly remind the reader of some terminology and background on valued and ordered fields (see [KS1] for more details).

Definition 1. Let K be a field and V be a K-vector space. Let Γ be a totally ordered set, and ∞ be an element larger than any element of Γ. A surjective map $v : V \to \Gamma \cup \{\infty\}$ is a *valuation* on V if for all $x, y \in V$ and $r \in K$, the following holds: (i) $v(x) = \infty$ if and only if $x = 0$, (ii) $v(rx) = v(x)$ if $r \neq 0$, (iii) $v(x - y) \geq \min\{v(x), v(y)\}$.

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An important example is when G is an ordered abelian group. Set $|g| := \max\{g, -g\}$ for $g \in G$. For non-zero $g_1, g_2 \in G$ say g_1 is archimedean equivalent to g_2 if there exists an integer r such that $r\,|g_1| \geq |g_2|$ and $r\,|g_2| \geq |g_1|$. Denote by $[g]$ the equivalence class of $g \neq 0$, and by v the *natural valuation* on G, that is, $v(g) := [g]$ for $g \neq 0$, and $v(0) := \infty$. If G is divisible, then G is a valued vector space over the rationals.

Definition 2. We say that $\{b_i : i \in I\} \subset V$ is *K-valuation independent* if for all $r_i \in K$ such that $r_i = 0$ for all but finitely many $i \in I$,

$$v\left(\sum_{i \in I} r_i b_i\right) = \min_{\{i \in I \,:\, r_i \neq 0\}} \{v(b_i)\}\ .$$

A *K-valuation basis* is a K-basis which is K-valuation independent.

We now recall some facts about valued fields (see [Ri] for more details).

Definition 3. Let K be a field, G an ordered abelian group and ∞ an element greater than every element of G.

A surjective map $w : K \to G \cup \{\infty\}$ is a *valuation* on K if for all $a, b \in K$ (i) $w(a) = \infty$ if and only if $a = 0$, (ii) $w(ab) = w(a) + w(b)$, (iii) $w(a - b) \geq \min\{w(a), w(b)\}$.

We say that (K, w) is a *valued field*. The *value group* of (K, w) is $w(K) := G$. The *valuation ring* of w is $\mathcal{O}_K := \{a : a \in K \text{ and } w(a) \geq 0\}$ and the *valuation ideal* is $\mathcal{I}(K) := \{a : a \in K \text{ and } w(a) > 0\}$. We denote by $\mathcal{U}(K)$ the multiplicative group $1 + \mathcal{I}(K)$ (the group of 1-units); it is a subgroup of the group of units (invertible elements) of $\mathcal{O}_K$. We denote by P the place associated to a valuation w; we denote the residue field by $KP = \mathcal{O}_K/\mathcal{I}(K)$. (We shall omit the K from the above notations whenever it is clear from the context.) For $b \in \mathcal{O}_K$, bP or bw is its image under the residue map. For a subfield E of K, we say that P is *E-rational* if P restricts to the identity on E and $KP = E$.

A valued field (K, w) is *henselian* if given a polynomial $p(x) \in \mathcal{O}[x]$, and $a \in Kw$ a simple root of the reduced polynomial $p(x)w \in Kw[x]$, we can find a root $b \in K$ of $p(x)$ such that $bw = a$.

There are important examples of valued fields. If $(K, +, \times, 0, 1, <)$ is an ordered field, we denote by v its natural valuation, that is, the *natural valuation* v on the ordered abelian group $(K, +, 0, <)$. (The set of archimedean

classes becomes an ordered abelian group by setting $[x]+[y]:=[xy]$.) Note that the residue field in this case is an archimedean ordered field, and that v is *compatible* with the order, that is, has a convex valuation ring.

Given an ordered abelian group G and a field F, denote by $F((G))$ the (generalized) *power series field* with coefficients in F and exponents in G; elements of $F((G))$ take the form $\sum_{g\in G} a_g t^g$ with $a_g \in F$ and well-ordered *support* $\{g \in G : a_g \neq 0\}$. We define the *minimal support valuation* on a non-zero element $f \in F((G))$ to be $v_{\min}(f) = \min \operatorname{support}(f)$. By convention, $v_{\min}(0) = \infty$.

Definition 4. Let E be a field and G an ordered abelian group. Given P a place on E, we define the ring homomorphism:

$$\varphi_P : \mathcal{O}_E((G)) \to (EP)((G)); \quad \sum_g a_g t^g \mapsto \sum_g (a_g P)\, t^g \,.$$

1.1 Motivation and Results

Brown in [B] proved that a valued vector space of countable dimension admits a valuation basis. This result was applied in [KS1] to show that every countable ordered field, henselian with respect to its natural valuation, admits a restricted exponential function, that is, an order preserving isomorphism from the ideal of infinitesimals $(\mathcal{I}(K), +, 0)$ onto the group of 1-units $(\mathcal{U}(K), \times, 1)$. We address the following question: *does every ordered field K, which is henselian with respect to its natural valuation, admit a restricted exponential function?* Let us consider the following illustrative example.

Example 5. **Puiseux series fields:** Let F be a real closed field. Then the function field $F(t)$ is an ordered field, where $0 < t < a$ for all $a \in F$. Define the real closed field of (generalized) Puiseux series over F to be

$$\mathrm{PSF}(F) = \bigcup_{n\in\mathbb{N}} F((t^{\frac{1}{n}}))\,,$$

and let $F(t)^\sim$ denote the real closure of $F(t)$. We then have the following containments of ordered fields:

$$F(t) \subset F(t)^\sim \subset \mathrm{PSF}(F) \subset F((\mathbb{Q}))\,.$$

Now, since F has characteristic 0, then the power series field $F((G))$ admits a $v_{\min}$-compatible restricted exponential exp with inverse log. These

are defined by

$$\exp(\varepsilon)=\sum_{i=0}^{\infty}\frac{\varepsilon^i}{i!}\quad\text{and}\quad\log(1+\varepsilon)=\sum_{i=1}^{\infty}(-1)^{i+1}\frac{\varepsilon^i}{i}\quad\text{where}\quad\varepsilon\in\mathcal{I}(F)\,.$$

(See [A].) The same argument as in the previous example shows that each term $F((t^{\frac{1}{n}}))$ in $\mathrm{PSF}(F)$ admits a restricted exponential. Therefore, so does $\mathrm{PSF}(F)$ itself. We now turn to the question of whether $F(t)^{\sim}$ admits a restricted exponential. Note that one could not just take the restriction of the exponential map exp defined above to the subfield $F(t)^{\sim}\subseteq F((\mathbb{Q}))$. Indeed, it can be shown that the map exp sends algebraic power series to transcendental power series, so the restriction of the exponential map exp to $F(t)^{\sim}$ is not even a well-defined map.

Following the strategy outlined at the beginning of this section, we shall instead investigate whether the multiplicative group of 1-units and the valuation ideal of $F(t)^{\sim}$ admit valuation bases.

It turns out that this question is interesting to ask for any valued field (not only for ordered valued fields):

Definition 6. Given a valued field (L,w), define a *w-restricted exponential* exp to be an isomorphism of groups between the valuation ideal of L and the 1-units of L (with respect to w) which is *w-compatible*; that is,

$$wa=w(1-\exp(a))\,.$$

The main results are Theorem 2.1 and Theorem 2.2 (see Section 2). We consider valued subfields L of a field of power series $F((G))$, where F is algebraically (or real) closed, and G is a countable ordered abelian group, which satisfy the *transcendence degree reduction property* (**TDRP**) over a countable ground field K (see Definitions 7 and 8; Section 2). We prove that the additive group of L admits a valuation basis as a K-valued vector space. In particular, the valuation ideal of L admits a valuation basis as a K-valued vector space. If the group of 1-units of L is divisible, we show that it admits a valuation basis over the rationals. We exhibit some interesting intermediate fields $F(G)\subseteq L\subseteq F((G))$ satisfying the TDRP over K. For instance, the field of rational functions $F(G)$ and the field $F(G)^{\sim}$ of power series in $F((G))$ algebraic over $F(G)$ satisfy it (see Theorem 3.7 and Theorem 3.8). We show that the class of fields satisfying the TDRP over K is closed under adjunction

of countably many elements of $K((G))$ — if L satisfies the TDRP over K, then so does $L(f_1, f_2, \ldots)$ (see Theorem 3.10).

In particular, if F is an archimedean ordered real closed field, and G is a countable divisible ordered abelian group, then the real closed field $F(G)^{\sim}$ admits a restricted exponential function. This gives a partial answer to the original question posed.

It is interesting to note that similar arguments are used in Section 11, p. 35 of [A-D] to show that certain ordered fields admit a derivation function.

The paper is organized as follows. In Section 2, we give a detailed statement of the main results. In Section 3, we work out several technical valuation theoretic results, needed for the proofs of the main results. In Section 3.2, we develop interesting tests to decide whether a generalized power series is rational, or algebraic over the field of rational functions. In Section 3.3, we discuss the TDRP in detail and prove Theorems 3.7, 3.8 and 3.10. Section 4 is devoted to the proofs of Theorems 2.1 and 2.2. Finally, in Section 5 we apply the results to ordered fields and to the complements of their valuation rings.

It turns out that by assuming $|F| \leq \aleph_1$, one can provide elementary proofs of Theorems 2.1 and 2.2 not requiring the technical machinery developed in Sections 3 and 4. We provide details in Appendix A (Theorems A.1 and A.2).

2 Main Results

In this paper, we will be particularly interested in subfields of $F((G))$ satisfying a certain closure property. We first provide a definition in the case where F is algebraically closed.

Definition 7 (TDRP — algebraic)**.** Let F be an algebraically closed field, K a countably infinite subfield of F and G a countable ordered abelian group. We say that an intermediate field L, for

$$F(G) \subseteq L \subseteq F((G))\,,$$

satisfies the *transcendence degree reduction property* (or TDRP) over K if:

1. whenever the intermediate field E, for $K \subseteq E \subseteq F$, is countable, then $E((G)) \cap L$ is countable; moreover, L is the colimit[1] of the $E((G)) \cap L$ over such E;

2. whenever $K \subseteq E \subset E' \subseteq F$ for algebraically closed intermediate fields E, E' and E'/E is a field extension of transcendence degree 1, then for finitely many power series $s_1, \ldots, s_n$ in $E'((G)) \cap L$, there exists an E-rational place P of E' such that $s_i \in \mathcal{O}_P((G))$ and $\varphi_P(s_i) \in E((G)) \cap L$ for all i;

3. for E, E', P as above, if $\{\alpha\}$ is a fixed transcendence basis of E'/E, we may assume that P sends α, α^{-1} to K.

The key point of the third axiom is that if P restricts to the identity on some intermediate field $K \subseteq K' \subseteq E'$ and is finite on some element c algebraic over $K'(\alpha)$, then cP is algebraic over K'. (See the proof of Proposition 3.4).

It turns out that many results for the real closed case are implied by those for the algebraically closed case; hence, we make the following analogous definition.

Definition 8 (TDRP — real algebraic)**.** Let F a real closed field, K a countably infinite subfield of F and G a countable ordered abelian group. We say that an intermediate field L, for

$$F(G) \subseteq L \subseteq F((G))$$

satisfies the *transcendence degree reduction property* over K if the intermediate field

$$F^a(G) \subseteq (L \oplus \sqrt{-1}L) \subseteq F^a((G))$$

does, where $F^a = F \oplus \sqrt{-1}F$ denotes the algebraic closure of F.

Note that an elementary argument from field theory shows that $F^a(G) = F(G) \oplus \sqrt{-1}F(G)$; we give an alternative argument in the proof of Theorem 3.7.

Consider an algebraically or real closed field F and a countable ordered abelian group G. We will exhibit later some interesting intermediate fields $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K. For instance, the field

[1] union over a directed set

of rational functions $F(G)$ and the field $F(G)^{\sim}$ of power series in $F((G))$ algebraic over $F(G)$ satisfy it. Moreover, the class of fields satisfying the TDRP over K is closed under adjunction of countably many elements of $K((G))$ — if L satisfies the TDRP over K, then so does $L(f_1, f_2, \ldots)$.

Remark 9. Note that $L(f_1, f_2, \ldots)$ doesn't necessarily have countable dimension over L, so we cannot resort to any generalization of Brown's theorem ([B]) in this situation.

Our primary objective of this paper is to prove the following result.

Theorem 2.1 (Additive). *Let F be an algebraically or real closed field, K a countably infinite subfield of F and G a countable ordered abelian group. If $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K, then the valued K-vector spaces $(L, +)$ and therefore $(\mathcal{I}(L), +)$ admit valuation bases.*

We also prove the following multiplicative analogue.

Theorem 2.2 (Multiplicative). *Let F be an algebraically or real closed field of characteristic zero, and G a countable ordered abelian group. If $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over $\mathbb{Q}$ and the group $(\mathcal{U}(L), \times)$ is divisible, then $(\mathcal{U}(L), \times)$ is a valued $\mathbb{Q}$-vector space and admits a $\mathbb{Q}$-valuation basis.*

Note that these results are trivial whenever F is assumed to be countable; by the TDRP axioms, L would be countable, and we could apply Brown's theorem ([B]). So, suppose F is uncountable. Our strategy then involves expressing uncountable objects, such as F, as the colimits of countable objects. In particular, suppose we express F as the colimit of countable subfields, say K_λ for indices λ in a directed set. (This is always possible; how we do it will depend whether we may assume $\operatorname{trdeg} F \leq \aleph_1$.) From this, it will follow that, in the additive situation, the group $\mathcal{I}(L)$ is the colimit of the countable groups $\mathcal{I}(K_\lambda((G)) \cap L)$; in the multiplicative situation, the group $\mathcal{U}(L)$ is the colimit of the countable groups $\mathcal{U}(K_\lambda((G)) \cap L)$.

We now restrict ourselves to the additive case; analogous remarks apply to the multiplicative case. Since each $\mathcal{I}(K_\lambda((G)) \cap L)$ is countable, we can find a valuation basis for it by Brown's theorem ([B]), say B_λ. If we are fortunate enough that these valuation bases are consistent in the sense that $B_{\lambda'}$ extends B_λ whenever $\lambda \prec \lambda'$, then we may take the colimit of the B_λ,

which will be our desired valuation basis of $\mathcal{I}(K_\lambda((G)) \cap L)$. How are we to choose the B_λ consistently? The answer lies in a generalization of Brown's theorem ([B]), featured as Corollary 3.6 in [KS2].

Definition 10. Let V/W be an extension of valued k-vector spaces with valuation w. For $a \in V$, we say that a has an *optimal approximation* in W if there exists $a' \in W$ such that for all $b \in W$, $w(a' - a) \geq w(b - a)$. We say that W has the *optimal approximation property* in V if every $a \in V$ has an optimal approximation in W.

The following proposition follows from Corollary 3.6 in [KS2]. (There, the term "nice" is used for the optimal approximation property.)

Proposition 2.3. *Let V/W be an extension of valued k-vector spaces. If W has the optimal approximation property in V and $\dim_k V/W$ is countable, then any k-valuation basis of W may be extended to one of V.*

We are then left show to show that $\mathcal{I}(K_\lambda((G)) \cap L)$ has the optimal approximation property in $\mathcal{I}(K_{\lambda'}((G)) \cap L)$ whenever $\lambda \prec \lambda'$; this will occupy the bulk of our arguments. Once we establish this, we are able to easily construct our desired valuation bases inductively.

We conclude with two remarks concerning the two main theorems.

Remark 11. Note that the assumption that $\operatorname{char} F = 0$ is necessary in Theorem 2.2. If $\operatorname{char} F = p$, then for any non-trivial element $f \in \mathcal{U}(L)$, we have $v_{\min}(1 - f^p) = p \cdot v_{\min}(1 - f) \neq v_{\min}(1 - f)$. Hence, $(\mathcal{U}(L), \times)$ does not admit a valued $\mathbb{Q}$-vector space structure, even if it is divisible.

Remark 12. Note that it can make a difference over which subfield we wish to take a valuation basis. By the results of this paper, we know that $\mathbb{R}(t)$ and $\mathbb{R}(t)^\sim$ both admit $\mathbb{Q}$-valuation bases. We claim they do not admit $\mathbb{R}$-valuation bases. Indeed, since $\mathbb{R}(t)$ and $\mathbb{R}(t)^\sim$ have residue field $\mathbb{R}$, if $\mathcal{B}$ is an $\mathbb{R}$-valuation independent subset, then the elements of $\mathcal{B}$ have pairwise distinct values. Therefore, $|\mathcal{B}| \leq |\mathbb{Q}| = \aleph_0$. On the other hand, the dimension of $\mathbb{R}(t)$, as a vector space over $\mathbb{R}$ is uncountable (e.g. the subset $\{(1 - xt)^{-1}\}_{x \in \mathbb{R}}$ is $\mathbb{R}$-linearly independent).

Concerning the choice of the ground field, we also record the following observation (which is of independent interest). The proof is straightforward, and we omit it.

Proposition 2.4. *Let V be a valued K-vector space and k be a subfield of K. If B denotes a K-valuation basis of V and B' denotes a k-vector space basis of K, then $B \otimes B' = \{b \otimes b' : b \in B, b' \in B'\}$ is a k-valuation basis of V.*

3 Technical results and key examples

We isolate here some results common to the proofs of our main theorems; note that the proofs of these results hold in every characteristic unless noted otherwise. As an application, we then give examples of fields satisfying the TDRP.

3.1 Constructing places

A basic tool in this paper will be the existence of certain places; these will often be used to decrease transcendence degree.

Proposition 3.1. *Consider a tower of fields*

$$K \subseteq E \subseteq E'$$

where K is infinite and E'/E is an extension of algebraically closed fields with transcendence basis $\{\alpha\}$. Suppose R is a subring of E' that is finitely generated over E. Then there exists an E-rational place P of E' such that the elements α and α^{-1} are sent to K and the place P is finite on R.

Proof. We assume without loss of generality that $\alpha, \alpha^{-1} \in R$; if not, simply adjoin them. We first exhibit a place of $\operatorname{Quot} R$ satisfying the stated conditions.

There are infinitely many E-rational places P of $\operatorname{Quot} R$ sending α and α^{-1} to K. Indeed, for each $q \in K$, we obtain the $(\alpha - q)$-adic place P_q on $E[\alpha]$ and therefore on $\operatorname{Quot} R$ by Chevalley's place extension theorem. Note that for $q \neq q'$, we necessarily have $P_q \neq P_{q'}$.

Moreover, we may select some q such that P_q is finite on R. For suppose $R = E[c_1, \ldots, c_n]$. Since the P_q are trivial on E, they are necessarily finite on any c_i algebraic over E. On the other hand, for any c_i transcendental over E, the $(1/c_i)$-adic place on $E(c_i)$ is the only one not finite on c_i; by extension, there are at most $[\operatorname{Quot}(R) : E(c_i)] < \infty$ places on $\operatorname{Quot}(R)$ not finite on c_i.

(Precisely how many depends on separability.) Since of the infinitely many places P_q only finitely many map c_i to ∞ for some i, we may fix a q such that P_q is finite on all c_i and thus finite on R.

Henceforth, write P to denote this place. By Chevalley's place extension theorem again, P on $\operatorname{Quot} R$ extends to a place on E' satisfying our desired properties. $\square$

Intuitively, the place P given by Proposition 3.1 is used to replace a field subextension of K in F of transcendence degree d by one of transcendence $d-1$. We may also make use of this tool for power series via the induced ring homomorphism φ_P. We now present a finiteness condition that enables us to apply this previous result.

Definition 13. Let (L, w) be a valued field. A *contraction* Φ on a subset S of L is a map $S \to S$ such that

$$w(\Phi a - \Phi b) > w(a - b) \text{ for all } a, b \in S\,.$$

Proposition 3.2. *For K a field and G an ordered abelian group, let $f \in K((G))$. Let f be algebraic over $K(G)$. If* $\operatorname{char}(K) = 0$*, then there exists a subring $R \subseteq K$ finitely generated over $\mathbb{Z}$ such that* $\operatorname{coeffs} f \subseteq R$*. If* $\operatorname{char}(K) = p > 0$*, then there exists a subring $R \subseteq K$ finitely generated over $\mathbb{F}_p$ such that* $\operatorname{coeffs} f \subseteq R^{1/p^\infty}$.

Proof. We prove a stronger result. Namely, let L be the algebraic closure of K, H be the divisible hull of G, v be the minimal support valuation $v_{\min}$. If $\operatorname{char}(K) = 0$, define

$$\begin{aligned} S \;=\; \{f \in L((H)) : \operatorname{coeffs} f \subset R \text{ for a subring } R \subseteq K \\ \text{finitely generated over } \mathbb{Z}\}\,, \end{aligned}$$

and if $\operatorname{char}(K) = p > 0$, define

$$\begin{aligned} S \;=\; \{f \in L((H)) : R^{1/p^\infty} \text{ contains } \operatorname{coeffs}(f) \text{ for a subring } R \\ \text{finitely generated over } \mathbb{F}_p\}\,. \end{aligned}$$

We show that S is an algebraically closed subfield of $L((H))$. For notational convenience, define $A = \mathbb{Z}$ if $\operatorname{char}(K) = 0$; otherwise, define $A = \mathbb{F}_p$.

We first establish that (S, v) is a henselian subfield. It is easily verified that S is in fact a field — for if $r, r' \in S$ are contained in finitely generated subrings

R, R', respectively, then $r - r'$ belongs to the finitely generated ring $A[R, R']$; if $r' \neq 0$, then r/r' belongs to the finitely generated ring $A[R, R', 1/c]$, where c is the leading coefficient of f'.

We now verify Hensel's Lemma. Take a monic polynomial $Q \in \mathcal{O}_S[t]$ and an approximate root $r \in \mathcal{O}_S$ such that $vQ(r) > 0$ and $vQ'(r) = 0$. Write $Q(t) = a_0 + a_1 t + \cdots + a_n t^n$, and let c be the leading coefficient of $Q'(r)$. We claim that r can be refined to a root f such that $\operatorname{coeffs} f \subseteq R$, where R is the ring $A[1/c, \operatorname{coeffs}(a_i, r)]$. By the Newton Approximation Method, we obtain a contraction:

$$\begin{aligned} \Phi : r + \mathcal{I}(\, R((G))\,) &\to r + \mathcal{I}(\, R((G))\,) \\ x &\mapsto x - Q(x)/Q'(r)\,. \end{aligned}$$

Since $\mathcal{I}(\, R((G))\,)$ is spherically complete, Φ has a fixed point, which is a root of Q in $r + \mathcal{I}(\, R((G))\,)$. Thus, S is henselian.

First assume that $\operatorname{char}(K) = 0$. Since the value group $vS = H$ is divisible and the residue field Sv is algebraically closed, it follows that S is algebraically closed. (See [P].)

Now assume that $\operatorname{char}(K) = p > 0$. The algebraic closure of S is a purely wild extension of S, and by Lemma 13.11 in [KF], S is equal to its own ramification field; by Theorem 7.15 in [KF] (which states that the ramification group is a pro-p group), if S is not algebraically closed, then we can find a separable extension S' of S of degree p. Such an extension S' is generated by an Artin-Schreier polynomial by Theorem 6.4 of [L]; however, this is impossible, since any root of an Artin-Schreier polynomial is once again contained in S.

Since S clearly contains $K(G)$, the desired result then follows; namely, that whenever f is algebraic over $K(G)$, then $f \in S$. $\square$

Note that in positive characteristic, the statement that $\operatorname{coeffs} f \subseteq R^{1/p^\infty}$ cannot be strengthened to $\operatorname{coeffs} f \subseteq R$. Indeed, let $K = \mathbb{F}_p(y)$ and $G = \mathbb{Q}$. Then the power series

$$f(t) = \sum_{i \geq 1} y^{1/p^i} t^{-1/p^i}$$

satisfies the relation $f^p - f - yt^{-1}$ and is therefore algebraic over $K(\mathbb{Q})$; on the other hand, the coefficient set of $f(t)$ is $\{y^{1/p^i}\}$, which is clearly not contained in any ring finitely generated over $K = \mathbb{F}_p(y)$.

We now apply these results to rational and algebraic series.

Proposition 3.3. *Let E'/E be an extension of algebraically closed fields with transcendence basis $\{\alpha\}$ and take an infinite subfield K of E. Given finitely many power series $s_1, \ldots, s_n \in E'(G) \subseteq E'((G))$, there exists an E-rational place P of E' sending α, α^{-1} to K such that $s_i \in \mathcal{O}_P((G))$ and $\varphi_P(s_i) \in E(G) \subseteq E((G))$ for each i.*

Proof. For each i, take $f_i, g_i \in E'[G]$ such that $s_i = f_i/g_i$; without loss of generality, assume that the g_i are monic. Observe that coeffs(s_i, f_i, g_i) are contained in the finitely generated ring $R[\text{coeffs}(f_i, g_i)]$; hence, by Proposition 3.1, there exists an E-rational place P of E' sending α, α^{-1} to K that is finite on R. Since each g_i is monic, the $\varphi_P(g_i)$ are non-zero; hence, $\varphi_P(s_i) = \varphi_P(f_i)/\varphi_P(g_i)$. □

Proposition 3.4. *Let E'/E be an extension of algebraically closed fields with transcendence basis $\{\alpha\}$ and take a infinite subfield K of E. Given finitely many power series $s_1, \ldots, s_n \in E'((G))$ that are algebraic over $E'(G)$, there exists an E-rational place P of E' sending α, α^{-1} to K such that $s_i \in \mathcal{O}_P((G))$ and $\varphi_P(s_i)$ is algebraic over $E(G)$ for each i.*

Proof. By Proposition 3.2, there exists a subring R of E', finitely generated over E, such that $\operatorname{coeffs} s_i \subseteq R$ (if $\operatorname{char} E = 0$) or $\operatorname{coeffs} s_i \subseteq R^{1/p^\infty}$ (if $\operatorname{char} E = p$) for each i. Pick a transcendence basis $\{\alpha\}$ of E'/E. Then, by Proposition 3.1, we may take an E-rational place P of E' that is finite on R, α and α^{-1} and sends α, α^{-1} to K.

Take s to be any of the s_i. As s is algebraic, suppose it is a root of the non-trivial polynomial $Q \in E[\alpha, t^g : g \in G][y]$. Notice that in the polynomial ring $E[\alpha]$, the kernel of P is the prime ideal $(\alpha - \alpha P)$. Since $E[\alpha]$ is a unique factorization domain, we may divide out coefficients of Q if necessary in order to assume that the polynomial $\varphi_P Q$ is non-zero. (In a slight abuse of notation, we extend φ_P to the polynomial ring over $\mathcal{O}_P((G))$.) As $\varphi_P s$ is a root of $\varphi_P Q \neq 0$, it is algebraic over $E(G)$, as desired. □

3.2 Coefficient tests for rational and algebraic power series

Using the results developed in the previous section, we can develop a simple coefficient test; in this section, G will denote an arbitrary ordered abelian group with no restrictions on its cardinality. For now, we make no assumptions about characteristic.

Proposition 3.5. *Let E/K be an extension of fields. Then,*

$$K((G)) \cap E(G) = K(G) \,.$$

Proof. It suffices to show that

$$K((G)) \cap E[G] = K[G] \,,$$

as the desired result follows by taking function fields of both sides. This is clear, since $f \in K((G)) \cap E[G]$ means that f has finite support with coefficients in K. $\square$

We have an algebraic power series analogue corresponding to Proposition 3.5.

Proposition 3.6. *Let E/K be an extension of fields. If E and K are both real closed or both algebraically closed, then*

$$K((G)) \cap E(G)^{\sim} = K(G)^{\sim} \,,$$

where $\cdot^{\sim}$ denotes relative algebraic closure in $E((G))$.

Proof. Letting H denote the divisible hull of G, we see that $E((H))$ is algebraically or real closed if E is algebraically or real closed, respectively. The inclusion "$\supseteq$" follows immediately.

To see the "$\subseteq$" inclusion, first assume that E, K are algebraically closed. Take a power series $s \in K((G)) \cap E(G)^{\sim}$; since s satisfies a polynomial relation in $E(G)$, we may assume that $\operatorname{trdeg} E/K$ is finite by replacing E by a subextension of K if necessary. Taking a filtration

$$K = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

where $\operatorname{trdeg} E_{i+1}/E_i = 1$ for all i, we apply Proposition 3.4 n times to see that $s \in K((G)) \cap E(G)^{\sim}$, as desired. If E, K are real closed, by reducing to the algebraically closed case, it suffices to note that any element f of $K((G))$ that is algebraic over $K^a((G))$ is also algebraic over $K((G))$. $\square$

3.3 TDRP for rational and algebraic power series

Fix an algebraically or real closed field F, a countably infinite subfield K and a countable ordered abelian group G. In this section, we exhibit some intermediate fields $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K.

Theorem 3.7. *Suppose that $L = F(G)$. Then, L satisfies the TDRP over K.*

Proof. Suppose that F is real closed; we give a reduction to the case when F is algebraically closed. Indeed, note that $F^a = F \oplus \sqrt{-1}F$, and we may take σ to be the non-trivial element of $\mathrm{Gal}(F^a/F)$. It is clear that $F(G) \oplus \sqrt{-1}F(G)$ is contained in $F^a(G)$. On the other hand, $F^a(G)$ is contained in $F(G) \oplus \sqrt{-1}F(G)$; for if h is the power series development of a rational function in $F^a(G)$, then $h = (h + \sigma(h))/2 + (h - \sigma(h))/2$ and these two summands are in $F(G)$ and $\sqrt{-1}F(G)$, respectively, by Proposition 3.5. Hence, $F(G) \oplus \sqrt{-1}F(G) = F^a(G)$, and it suffices to prove the theorem when F is algebraically closed by definition of the TDRP.

Thus, assume that F is algebraically closed. The first condition of the TDRP is obvious — if E is a field extension of K and E is countable, then $E((G)) \cap L = E(G)$ (with equality from Proposition 3.5) is countable. The second and third conditions are simply the statement of Proposition 3.3. $\square$

Theorem 3.8. *Suppose that $L = F(G)^\sim$, the relative algebraic closure of $F(G)$ in $F((G))$. Then, L satisfies the TDRP over K.*

Proof. As above, we may assume that F is algebraically closed after verifying that $F(G)^\sim \oplus \sqrt{-1}F(G)^\sim = F^a(G)^\sim$, where $\cdot^\sim$ denotes relative algebraic closure in $F((G))$ for the first two instances and in $F^a((G))$ for the third. Note that any element f of $K((G))$ that is algebraic over $K^a((G))$ is also algebraic over $K((G))$. Verification of the TDRP properties proceeds nearly identically; for the second condition of the TDRP, use Proposition 3.4 instead of 3.3. $\square$

We now show that the class of fields satisfying the TDRP over K is closed under the adjunction of countably many power series in $K((G))$.

Lemma 3.9. *Suppose that the intermediate field $F(G) \subseteq L \subseteq F((G))$ satisfies the TDRP over K, where F is algebraically closed. Consider an algebraically closed and countable subextension $K \subseteq E \subseteq F$. Then, for any power series $h \in K((G))$, we have*

$$L(h) \cap E((G)) = (E((G)) \cap L)(h)\,.$$

Proof. We show the "$\subseteq$" direction; the other is clear.

Suppose that $s \in L(h) \cap E((G))$; we may take some countable algebraically closed field E' containing E such that

$$s = (f_0 + f_1 h + \cdots + f_n h^n)/(g_0 + g_1 h + \cdots + g_m h^m)$$

for $f_i, g_i \in L \cap E'((G))$. If h is algebraic over L, we may assume the denominator above is 1; otherwise, we may assume that $g_0 = 1$. Without loss of generality, we may take a chain $E = E_0 \subset E_1 \subset \cdots \subset E_n = E'$ of algebraically closed intermediate fields E_i such that the E_{i+1}/E_i are extensions of transcendence degree 1. By applying the second property of the TDRP n times to the displayed equation above, the first statement follows; note that our assumption on the denominator implies that it does not vanish. $\square$

Theorem 3.10. *Suppose that the intermediate field $F(G) \subseteq L \subseteq F((G))$ satisfies the TDRP over K. Then if $\{h_i\}_{i\geq 1}$ are power series in $K((G))$, the field $L(h_i : i \geq 1)$ also satisfies the TDRP over K.*

Proof. As usual, it suffices to prove the result when F is algebraically closed. Indeed, suppose that $\{h_i\}_{i\geq 1}$ are power series in $K((G))$. Then, it is easily shown that

$$L(h_i : i \geq 1) \oplus \sqrt{-1}L(h_i : i \geq 1) = (L \oplus \sqrt{-1}L)(h_i : i \geq 1)\,;$$

the definition of TDRP for the algebraically closed field F^a then applies.

Henceforth, suppose F is algebraically closed. To simplify notation, we will denote $E((G)) \cap L$ by L_E for any field E. It suffices to verify the second condition of the TDRP, the rest being trivial. Furthermore, it suffices to show that if L satisfies the TDRP over K, then so does $L(h)$ — given finitely many power series $s_1, \ldots, s_n$ in

$$L(h_i : i \geq 1) \cap E((G)) = L_E(h_i : i \geq 1)$$

(with equality from Lemma 3.9), we may select finitely many $h_1, \ldots, h_m$ (after reindexing) such that $s_1, \ldots, s_n \in L(h_1, \ldots, h_m)$.

We proceed with the proof. Let E, E' be algebraically closed fields and E'/E an extension of transcendence degree 1. Given $s_1, \ldots, s_n$ in $E'((G)) \cap L(h)$, we may write $s_i = S_i(h)/Q_i(h)$ for polynomials $S_i(x), Q_i(x)$ in $L[x]$ by Lemma 3.9. Moreover, if h is algebraic, we assume the Q_i are constant; otherwise, assume that each Q_i is monic.

Since L satisfies the TDRP over K, we may take an E-rational place P of E' such that $\text{coeffs}(S_i, Q_i) \subseteq \mathcal{O}_P((G))$ and $\varphi_P(\text{coeffs}(S_i, Q_i)) \subseteq L_E$. (Observe that since S_i, Q_i are considered polynomials in $L[h]$, their coefficients lie in $L \subseteq F((G))$.) Extending φ_P to polynomials over $L_{E'}$, we see that $\varphi_P(S_i), \varphi_P(Q_i)$ are polynomials over L_E; hence, $\varphi_P(S_i)(h), \varphi_P(Q_i)(h)$ are elements oF $L_E(h)$. Recall that if h is algebraic over L, then the Q_i are constant; otherwise, they are monic. Hence, the $\varphi_P(Q_i)(h)$ are non-zero and therefore $\varphi_P(s_i) \in L_E$ for all i, as desired. $\square$

Remark 14. Since G is countable, there exists a countable extension field of K containing the coefficients of countably many power series in $K((G))$. In particular, this means that if $L \subseteq F((G))$ satisfies the TDRP over K, then for countably many power series $(h_i)_{i \geq 1}$ in $F((G))$, there exists a countable extension field K'/K such that $L(h_i : i \geq 1)$ satisfies the TDRP over K'.

4 Constructing valuation bases via TDRP

In this section, we seek out to prove Theorems 2.1 and 2.2. In what follows, F denotes an algebraically or real closed field, and we consider a countable subfield $K \subset F$.

Our strategy is to express F as the colimit of countable subfields of finite transcendence degree over K. More precisely, fix a transcendence basis $\{\alpha_\lambda\}_{\lambda \in I}$ of F over K. Notice that the family of finite subsets of I forms a directed set under inclusion — for each such finite subset $X \subset I$, define the subfield

$$K_X = K(\alpha_\lambda : \lambda \in X)^\sim \subseteq F\,,$$

where $\cdot^\sim$ denotes relative algebraic closure in F. Observe that just as $\varinjlim X = I$, $\varinjlim K_X = F$. Moreover, by the first TDRP axiom,

$$\begin{aligned} \varinjlim K_X((G)) \cap L &= L\,, \\ \varinjlim \mathcal{I}(K_X((G)) \cap L) &= \mathcal{I}(L) \quad \text{and} \\ \varinjlim \mathcal{U}(K_X((G)) \cap L) &= \mathcal{U}(L)\,. \end{aligned}$$

Given any finite subset X of I, we will need the optimal approximation property for the valued vector space extensions

$$\langle \mathcal{I}(K_Y((G)) \cap L) : Y \subset X \rangle \subseteq \mathcal{I}(K_X((G)) \cap L)\,.$$

Consequently, we will fix X throughout this section. For notational convenience, label the elements of X to be $x_1, x_2, \dots, x_N$, so that

$$X = \{x_1, x_2, \dots, x_N\}\,;$$

for $1 \le i \le N$, let $Y_i = X \setminus \{x_i\}$ and $Y_{i,j} = X \setminus \{x_i, x_j\}$.

Our desired results in the case that F is real closed will follow from the corresponding results when F is algebraically closed. Hence, we will assume that F is algebraically closed for now.

4.1 Complements of valuation rings in characteristic 0

The relevance of this section to the rest of the paper is to establish Theorem 4.3 in the sequel; the second half of this section is technically unnecessary and is provided for the sake of independent interest and perspective.

Out of necessity, $\operatorname{char} F = 0$ throughout. For simplicity, we assume also that F is algebraically closed.

Suppose that we have a K_{Y_N}-rational place P of K_X sending α_N, α_N^{-1} to K. Consider a sum a of elements of the K_Y for $Y \subset X$; that is,

$$a = a_1 + a_2 + \cdots + a_N \text{ for } a_i \in K_{Y_i}\,.$$

We would like to show that whenever aP is finite, we may assume that we also have a representation of the form

$$a = b_1 + b_2 + \cdots + b_N \text{ for } b_i \in K_{Y_i}\,,$$

where each b_iP is finite.

Note that $\mathcal{O}_{K_X}$ is a $\overline{K_X}$-vector space, where $\overline{K_X}$ denotes the residue field of K_X, so there exists a $\overline{K_X}$-vector space complement C of $\mathcal{O}_{K_X}$ in K_X; that is, $K_X = C \oplus \mathcal{O}_{K_X}$. Observe that for each $1 \le i \le k$,

$$(K_{Y_i} \cap C) \oplus (K_{Y_i} \cap \mathcal{O}_{K_X}) \subseteq K_{Y_i}\,.$$

Assuming equality held in the equation above, we could uniquely write $a_i = b_i + c_i$ for $b_i \in \mathcal{O}_{K_{Y_i}}$ and $c_i \in C$ — note that $\mathcal{O}_{K_{Y_i}} = \mathcal{O}_{K_X} \cap K_{Y_i}$. Our immediate aim is therefore to construct such a complement C where such equality in fact holds.

Lemma 4.1. *Suppose that F is algebraically closed and P is a K_{Y_N}-rational place of K_X sending α_N, α_N^{-1} to K. Then, there exists a complement C of $\mathcal{O}_{K_X}$ in K_X such that for each $1 \leq i \leq N$, $C \cap K_{Y_i}$ is a complement of $\mathcal{O}_{K_{Y_i}} = \mathcal{O}_{K_X} \cap K_{Y_i}$ in K_{Y_i}.*

Proof. Let $\mathrm{PSF}(\overline{K_X})$ denote the field of Puiseux series over $\overline{K_X}$; that is,

$$\mathrm{PSF}(\overline{K_X}) = \bigcup_{n=1}^{\infty} \overline{K_X}((t^{1/n}))\,.$$

We consider $\mathrm{PSF}(\overline{K_X})$ to be a valued field with the minimal support valuation $v_{\min}$. Since the residue field $\overline{K_X}$ is algebraically closed and of characteristic 0, it is well-known that $\mathrm{PSF}(\overline{K_X})$ is algebraically closed.

Note that since we consider F to be algebraically closed, we have that $\overline{K_X} = K_{Y_N}$. As $\alpha_{x_N}P, \alpha_{x_N}^{-1}P \in K$ by construction, we see that the element $\beta = \alpha_{x_N} - \alpha_{x_N}P \in K_{\{x_N\}}$ is transcendental over $\overline{K_X} = K_{Y_N}$; note that $\beta P = 0$. We thus define the embedding

$$\iota : K_{Y_N}(\beta) \to \mathrm{PSF}(\overline{K_X})$$

such that ι restricts to the identity on K_{Y_N} and sends β to t. Since $\beta P = 0$, we have that ι preserves the valuation v_P on $K_{Y_N}[\beta]$; it follows that it does so on $K_{Y_N}(\beta)$ as well. Another easy consequence of $\beta P = 0$ is that $\mathcal{O}_{K_{Y_i}} = K_{Y_{i,N}}$; this is proved as was Proposition 3.4.

Since $\mathrm{PSF}(\overline{K_X})$ is algebraically closed and K_X is an algebraic field extension of $K_{Y_N}(\beta)$, ι extends to an embedding:

$$\iota : K_X \to \mathrm{PSF}(\overline{K_X})\,.$$

Note that this induces a valuation $w = v_{\min} \circ \iota$ on K_X. We may assume without loss of generality that $w = v_P$; for K_X is algebraic over $K_{Y_N}(\beta)$, and therefore there exists $\sigma \in \mathrm{Gal}(K_X/K_{Y_N}(\beta))$ such that $w \circ \sigma = v_P$. Thus, if we consider instead the embedding $\iota' = \iota \circ \sigma$, we have that ι' preserves valuations; that is, $v_P = v_{\min} \circ \iota'$.

Moreover, for each $1 \leq i \leq N$, we have that

$$\iota(K_{Y_i}) \subseteq \mathrm{PSF}(\overline{K_{Y_i}})\,.$$

Note that this is immediate for $i = N$, as ι restricts to the identity on K_{Y_N}. For $i \neq N$, notice that K_{Y_i} is algebraic over $K_{Y_{i,N}}(\alpha_{x_N})$; moreover, ι

restricted to $K_{Y_{i,N}} \subset K_{Y_N}$ is the identity. Consequently, $\iota(K_{Y_i})$ is algebraic over $K_{Y_{i,N}}(\iota(\alpha_{x_N}))$. Since $\alpha_{x_N} = \beta + \alpha_{x_N} P$, we have $\iota(\alpha_{x_N}) = t + \alpha_{x_N} P$; this implies that $\iota(K_{Y_{i,N}}(\alpha_{x_N}))$ and, by algebraicity, $\iota(K_{Y_i})$ are contained in $\mathrm{PSF}(\overline{K_{Y_i}})$.

We are ready to construct our complement of C as stated. In the following displays, S will denote a finite negative subset of $\mathbb{Q}$. Note first that

$$C_P = \{\sum_{q \in S} c_q t^q : c_q \in \overline{K_X} \text{ and } S \text{ is a finite negative subset of } \mathbb{Q}\}$$

is a complement to the valuation ring of $\mathrm{PSF}(\overline{K_X})$. Moreover, since it is contained in the image of ι and ι preserves value, we deduce that $\iota^{-1}(C_P)$ is a complement of $\mathcal{O}_{K_X}$. That is, if

$$C = \iota^{-1}(C_P) = \{\sum_{q \in S} c_q \beta^q : c_q \in \overline{K_X} \text{ for some } S\}\,,$$

then

$$K_X = C \oplus \mathcal{O}_{K_X}\,.$$

It remains to verify that for each $1 \leq i \leq N$,

$$K_{Y_i} \subseteq (K_{Y_i} \cap C) \oplus (K_{Y_i} \cap \mathcal{O}_{K_X})\,.$$

Note that for $i = N$, this follows immediately, for $K_{Y_N} \subseteq \mathcal{O}_{K_X}$. For other i, the fact that $\iota(K_{Y_i}) \subseteq \mathrm{PSF}(\overline{K_{Y_i}})$ (and $\mathcal{O}_{K_{Y_i}} = K_{Y_{i,N}}$) shows that

$$\iota(K_{Y_i}) \cap C_P = \{\sum_{q \in S} c_q t^q : c_q \in \overline{K_{Y_i}} \text{ for some } S\}\,,$$

which is a complement of the valuation ring $\mathcal{O}_{\iota(K_{Y_i})}$ in $\iota(K_{Y_i})$. Pulling back by the value-preserving embedding ι, it follows that

$$\iota^{-1}(\iota(K_{Y_i}) \cap C_P) = K_{Y_i} \cap C = \{\sum_{q \in S} c_q \beta^q : c_q \in K_{Y_{i,N}}\}$$

is a complement to $\mathcal{O}_{K_{Y_i}}$ in K_{Y_i}, where $Y_{i,N} = Y_i \setminus \{x_N\}$; that is,

$$K_{Y_i} = (K_{Y_i} \cap C) \oplus \mathcal{O}_{K_{Y_i}} = (K_{Y_i} \cap C) \oplus (K_{Y_i} \cap \mathcal{O}_{K_X})\,.$$

$\square$

It is still possible to prove the previous result in the case that F is real closed; however, significantly more work is needed to eliminate negative parts of the power series given by ι in the proof above. We do not provide details here, as it suffices to consider the case that F is algebraically closed for now.

We can now construct complements as suggested from the beginning of this section.

Lemma 4.2. *Let $\langle K_Y : Y \subset X \rangle$ denote the additive subgroup of K_X generated by the subgroups K_Y and suppose that P is a K_{Y_N}-rational place of K_X sending $\alpha_{x_N}, \alpha_{x_N}^{-1}$ to K. Then, with respect to the place P,*

$$\mathcal{O}_{\langle K_Y : Y \subset X \rangle} = \langle \mathcal{O}_{K_Y} : Y \subset X \rangle \ .$$

More precisely,

$$\mathcal{O}_{\langle K_Y : \{N\} \subseteq Y \subset X \rangle} = \langle \mathcal{O}_{K_Y} : Y \subset X \rangle \ .$$

Proof. It suffices to show the "$\subseteq$" direction; the other is immediate. Suppose that $a \in \mathcal{O}_{\langle K_Y : Y \subset X \rangle}$. We may write $a = a_1 + a_2 + \cdots + a_N$ where $a_i \in K_{Y_i}$ for proper subsets $Y_i \subset X$. (Recall that Y_i was defined to be $X \setminus \{x_i\}$, where $X = \{x_1, x_2, \ldots, x_N\}$.)

By Lemma 4.1, we may take a decomposition $K_X = C \oplus \mathcal{O}_{K_X}$ such that for each $1 \leq i \leq N$,

$$K_{Y_i} = (K_{Y_i} \cap C) \oplus (K_{Y_i} \cap \mathcal{O}_{K_X}) \ .$$

Accordingly, we thus write the a_i as $b_i + c_i$, for $b_i \in \mathcal{O}_{K_{Y_i}}$ and $c_i \in C$ — note that $\mathcal{O}_{K_{Y_i}} = \mathcal{O}_{K_X} \cap K_{Y_i}$. Since $a = \sum b_i + \sum c_i$ is in $\mathcal{O}_{K_X}$, it follows that $\sum c_i = 0$; that is, $a = b_1 + b_2 + \cdots + b_k$. Noting that $c_N = 0$ and therefore $a_N = b_N$, both claims now follow. $\square$

Note that in positive characteristic, we can no longer assume that there exist complements as given in Lemma 4.1; the proof fails as we can no longer assume that the negative part of the support of an algebraic power series, and particularly of an element in the image of ι, is finite. (For example, see the remarks following Proposition 3.2.) See Theorem 4.3 in the next section for a weakened version of Lemma 4.2 that holds independently of $\operatorname{char} F$.

4.2 Output of places

Our later combinatorial arguments will depend on a cancellation property of a ring homomorphism φ_P implied by the result here.

As before, we assume K is countably infinite; this is in order that Proposition 3.1 holds. The following lemma is a weak analogue to Lemma 4.2 that holds independently of $\operatorname{char} F$; if $\operatorname{char} F = 0$, it follows as an immediate corollary.

Note that since F is assumed to be algebraically closed, $K_X P = K_{Y_N}$.

Theorem 4.3. *Let P be a K_{Y_N}-rational place of K_X sending $\alpha_{x_N}, \alpha_{x_N}^{-1}$ to K. Suppose that $a \in \mathcal{O}_{\langle K_Y : Y \subset X \rangle}$; that is, $a = a_1 + a_2 + \cdots + a_k$ for $a_i \in K_{Y_i}$ and P is finite on a. Then $aP \in \langle K_Y : Y \subset X \rangle$; that is, $aP = b_1 + b_2 + \cdots + b_k$ for $b_i \in K_{Y_i}$.*

Moreover, we may assume that $b_N = a_N$ and that for any i such that $a_i = 0$, then $b_i = 0$.

Proof. Take a field embedding ι as in the proof of Lemma 4.1, and define b_i to be the constant term of $\iota(a_i)$; that is, $b_i = 0(\iota(f))$. □

4.3 The optimal approximation property

Consider an intermediate field $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K. We give a combinatorial formula for an optimal approximation h to a power series $f \in L$ in terms of ring homomorphisms φ_P given by the second axiom of the TDRP. Since it will follow that $s \in L$ as well, this conceptually means that the field L is "closed under taking optimal approximations."

Theorem 4.4. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is algebraically closed. Take $\langle\cdot\rangle$ in the context of additive groups. If $f \in K_X((G)) \cap L$, then there exists for each $1 \le i \le N$ a K_{Y_i}-rational place P_i of K_X such that*

$$h = f - (\mathrm{id} - \varphi_{P_1}) \circ \cdots \circ (\mathrm{id} - \varphi_{P_N}) f$$

is an element of $\langle K_Y((G)) \cap L : Y \subset X \rangle$ and an optimal approximation to f in $\langle K_Y((G)) : Y \subset X \rangle$; the respective statement holds for $\mathcal{I}(K_X((G)))$.

Proof. It suffices to prove the first statement.

We define the places P_k by decreasing induction on k from N to 1. For notational ease, whenever the places P_i have been defined for all $k < i \le N$, we define

$$f_k = (\mathrm{id} - \varphi_{P_{k+1}}) \circ \cdots \circ (\mathrm{id} - \varphi_{P_N})\, f\,;$$

moreover, for any $(N-k+1)$-tuple $\sigma = (e_k, e_{k+1}, \ldots, e_N)$ over $\mathbb{F}_2$, we define

$$f_\sigma = \psi^\sigma(f)\,, \quad \text{where } \psi^\sigma = (-\varphi_{P_k})^{e_k} \circ \cdots \circ (-\varphi_{P_N})^{e_N}\,.$$

(For any function φ, we let $\varphi^1 = \varphi$ and $\varphi^0 = \mathrm{id}$.) Observe that this means $f_N = f = f_{()}$, which is in $K_X((G)) \cap L$.

Suppose that for some k, P_i has been defined for all $k < i \leq N$ and that f_k is a power series in $K_X((G)) \cap L$. Then by the second TDRP axiom, we may take a K_{Y_k}-rational place P_k of K_X such that for all $(N-k+1)$-tuples σ over $\mathbb{F}_2$, P is finite on coeffs(f_k, f_σ) and $\varphi_P(f_k)$, $\varphi_P(f_\sigma)$ are power series in $K_{Y_k}((G)) \cap L$. Note that we then have $f_{k-1} \in K_X((G)) \cap L$, as desired.

Having defined our places, we check our two properties hold. Let $\sigma = (e_1, \ldots, e_N)$ denote a non-zero tuple. If i denotes the least index such that $e_i = 1$, then $f_\sigma \in K_{Y_i}((G)) \cap L$ by the third TDRP axiom. Since

$$-h = \sum f_\sigma\,,$$

the sum over non-zero N-tuples σ, we see that $h \in \langle K_Y((G)) \cap L : Y \subset X \rangle$, as desired.

To see that h is an optimal approximation to f in $\langle K_Y((G)) : Y \subset X \rangle$, it suffices to show that if $\alpha(f) \in \langle K_Y : Y \subset X \rangle$ for some exponent α, then $\alpha(h) = \alpha(f)$. Indeed, for such an α, write

$$\alpha(f) = a_1 + a_2 + \cdots + a_N \quad \text{where } a_i \in K_{Y_i}\,;$$

by decreasing induction on k using Theorem 4.3, we may write

$$\alpha(f_k) = b_1 + b_2 + \cdots + b_k \quad \text{where } a_i \in K_{Y_i}\,.$$

Hence, $\alpha(h) = \alpha(f - f_0) = \alpha(f)$, as desired. □

We have a multiplicative analogue.

Theorem 4.5. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is algebraically closed and of characteristic 0. Take $\langle \cdot \rangle$ in the context of multiplicative groups. If $f \in \mathcal{U}(K_X((G)) \cap L)$, then there exists for each $1 \leq i \leq k$ a K_{Y_i}-rational place P_i of K_X such that*

$$h = f \,/\, (\mathrm{id} \,/\, \varphi_{P_1}) \circ \cdots \circ (\mathrm{id} \,/\, \varphi_{P_k})\, f$$

is an element of $\langle \mathcal{U}(K_Y((G)) \cap L) : Y \subset X \rangle$ and an optimal approximation to f in $\langle \mathcal{U}(K_Y((G))) : Y \subset X \rangle$.

Proof. The construction of the places P_k proceeds identically as in Theorem 4.4. Verification of two stated properties is a straightforward modification from before, after recalling that the map exp introduced in Example 5 is a group isomorphism from $(\mathcal{I}(F((G))), +)$ to $(\mathcal{U}(F((G))), \times)$ with inverse log such that $v_{\min}(1-\exp(f)) = v_{\min}(f)$. Moreover, note that the maps exp and log commute with the ring homomorphism φ_P.

In particular, to check that each f_σ, for σ a non-zero tuple, is contained in some $K_Y((G))$, simply note that

$$f_\sigma = \psi^\sigma(f) = (\exp \circ \psi^\sigma \circ \log)(f)\,.$$

Similarly, h is an optimal approximation to f in $\langle \mathcal{U}(K_Y((G))) : Y \subset X \rangle$ if and only if $\log(h)$ is an optimal one to $\log(f)$ in $\langle \mathcal{I}(K_Y((G))) : Y \subset X \rangle$. $\square$

Our optimal approximation results that we will use to extend valuation bases now follow immediately; note that we now also consider the case when F is real closed.

Theorem 4.6. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is a real closed or algebraically closed field. Take $\langle \cdot \rangle$ in the context of additive groups. Then,*

$$\langle \mathcal{I}(K_Y((G)) \cap L) : Y \subset X \rangle$$

has the optimal approximation property in $\mathcal{I}(K_X((G)) \cap L)$.

Proof. If F is algebraically closed, then this is deduced immediately from Theorem 4.4. Otherwise, if F is real closed, we reduce to the algebraically closed case. In particular, given an element $f \in \mathcal{I}(K_X((G)) \cap L)$, we may regard f as an element of $(L \oplus \sqrt{-1}L) \cap K_X^a((G))$. By definition, $(L \oplus \sqrt{-1}L)$ is a subfield of the algebraically closed field F^a and satisfies the TDRP over K; applying the desired result, we obtain an optimal approximation s to f in

$$\left\langle (L \oplus \sqrt{-1}L) \cap K_Y^a((G)) : Y \subset X \right\rangle\,.$$

Taking σ to be the non-trivial element of $\mathrm{Gal}((L \oplus \sqrt{-1}L)/L)$, sending $a + \sqrt{-1}b$ to $a - \sqrt{-1}b$ for $a, b \in L$, we see that $(s + \sigma(s))/2$ is an optimal approximation to f in $\langle K_Y((G)) \cap L : Y \subset X \rangle$, as desired. $\square$

Theorem 4.7. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is a real closed or algebraically closed field of characteristic* 0. *Take $\langle\cdot\rangle$ in the context of multiplicative groups. Then,*

$$\langle \mathcal{U}(K_Y((G)) \cap L) : Y \subset X \rangle$$

has the optimal approximation property in $\mathcal{U}(K_X((G)) \cap L)$.

Proof. As above. □

4.4 Extending valuation bases

Using our optimal approximation results, we can now exhibit valuation bases for $\mathcal{I}(L, +)$ and $\mathcal{U}(L, \times)$, where L is a subfield of F satisfying the TDRP over K. (Note that when $\operatorname{char} K > 0$, only the additive case applies.) Recall that we have chosen a transcendence basis $\{\alpha_\lambda\}_{\lambda \in I}$ of F over K, and for each finite subset X of I, we have $K_X = K(\alpha_\lambda : \lambda \in X)^\sim$.

For each X, let V_X denote the valued K-vector space $\mathcal{I}(K_X((G)) \cap L)$. If $\mathcal{U}(K_X((G)) \cap L)$ is a divisible group, let W_X denote the valued $\mathbb{Q}$-vector space $\mathcal{U}(K_X((G)) \cap L)$.

For successively larger n, our aim is to define a valuation basis B_X for each valued vector space V_X (or W_X in the multiplicative case) where $|X| = n$, extending the valuation bases B_X for $|X| < n$. We first give a lemma assuring that the valuation bases B_X for $|X| = n$ can be chosen independently, as long as they extend the valuation bases B_Y for $Y \subset X$.

Lemma 4.8. *Let $\langle\cdot\rangle$ denote K-vector space span. For a finite subset $X \subseteq I$,*

$$K_X \cap \langle K_Z : Z \not\supseteq X \text{ finite}\rangle = \langle K_Y : Y \subset X\rangle \ .$$

Proof. We first assume that F is algebraically closed. Let $Z_1, \ldots, Z_k$ be finite subsets of the index set I not containing the subset X, and suppose that $y = y_1 + \cdots + y_k \in K_X$. It then suffices to show $y \in \langle K_Y : Y \subset X\rangle$.

Writing $Z = X \cup Z_1 \cup \cdots \cup Z_k$, we see that K_Z has finite transcendence degree over K_X. Hence, we may take a chain of algebraically closed fields

$$K_X = E_0 \subset E_1 \subset \cdots \subset E_n = K_Z$$

where each field extension E_{i+1}/E_i is of transcendence degree 1. By repeated application of Proposition 3.1, for decreasing values of i from $n-1$ to 0, we can

take an E_i-rational place P_i of E_{i+1} that is finite on the $y_j P_{i+1} P_{i+2} \cdots P_{n-1}$ and sends the transcendence basis of E_{i+1} over E_i to K.

By repeated application of Proposition 3.4, $y_j P \in K_{X \cap Z_j}$ for all $1 \leq j \leq k$, where we write P to denote the composition of places $P_0 P_1 \cdots P_{n-1}$. Since $X \cap Z_j$ is a proper subset of X, we have

$$\begin{aligned} y &= yP \\ &= y_1 P + \cdots + y_k P \\ &\in \langle K_{X \cap Z_1}, \ldots, K_{X \cap Z_k} \rangle \\ &\subseteq \langle K_Y : Y \subset X \rangle \ . \end{aligned}$$

Now if F is real closed, then taking algebraic closures and applying the result in the algebraically closed case, we see that

$$K_X^a \cap \langle K_Z^a : Z \not\supseteq X \text{ finite} \rangle = \langle K_Y^a : Y \subset X \rangle \ ,$$

from which the desired result follows immediately. ☐

Theorem 4.9. *Let $n \geq 0$, and suppose that for each subset X of I of cardinality at most n, we have a valuation basis B_X of*

$$V_X = \mathcal{I}(\, K_X((G)) \cap L\,)\,.$$

Suppose that

$$\mathcal{B}_n = \bigcup (B_X : X \subseteq I, |X| \leq n)$$

is valuation independent. Then, for each subset X of I of cardinality $n+1$, we may define a valuation basis B_X of V_X such that

$$\mathcal{B}_{n+1} = \bigcup (B_X : X \subseteq I, |X| \leq n+1)$$

is valuation independent.

Proof. Observe that since $\mathcal{B}_n$ is valuation independent, B must be inclusion-preserving. Indeed, suppose $X' \subset X$ of cardinality at most n. By assumption, $B_{X'} \cup B_X$ is valuation independent and therefore a valuation basis of V_X. Since B_X is a valuation basis of V_X and therefore maximally valuation independent, we must have $B_{X'} \cup B_X = B_X$ and $B_{X'} \subset B_X$.

We now define a valuation basis B_X of V_X for each subset X of I of cardinality $n+1$. For such a subset X, observe that $\bigcup (B_Y : Y \subset X)$ is a valuation

basis of $\langle V_Y : Y \subset X\rangle$. By Theorem 4.6, the subspace $\langle V_Y : Y \subset X\rangle$ has the optimal approximation property in V_X; moreover, since V_X is countable, it has countable dimension over $\langle V_Y : Y \subset X\rangle$. Therefore, Proposition 2.3 allows us to extend $\bigcup(B_Y : Y \subset X)$ to a valuation basis of V_X, and we define B_X to be this.

It remains to show that $\mathcal{B}_{n+1}$ is valuation independent. Consider a finite sum

$$a = c_1b_1 + c_2b_2 + \cdots + c_kb_k$$

for non-zero scalars $c_i \in K$ and distinct elements $b_i \in \mathcal{B}_{n+1}$ such that $q = v_{\min}(v_1) = v_{\min}(v_2) = \cdots = v_{\min}(v_k)$. Since we know $\mathcal{B}_n$ is valuation independent, we may assume with loss of generality (by reindexing if necessary) that there exists some subset $X \subset I$ of cardinality $(n+1)$ and an index $1 \leq j \leq k$ such that

$$b_i \in B_X \setminus \bigcup(B_Y : Y \subset X)$$

if and only if $i \leq j$. Since B_X is valuation independent, the coefficient $q(c_1v_1 + c_2v_2 + \cdots + c_jv_j)$ is in $K_X \setminus \langle K_Y : Y \subset X\rangle$. As the coefficient $q(c_{j+1}v_{j+1} + c_{j+2}v_{j+2} + \cdots + c_kv_k)$ is clearly in $\langle K_Z : Z \neq X, |Z| \leq n+1\rangle$, Lemma 4.8 implies that $q(a) \neq 0$. Hence, $v_{\min}(a) = q$, and $\mathcal{B}_{n+1}$ is valuation independent. □

In the case char $F = 0$, we have a multiplicative analogue.

Theorem 4.10. *Let $n \geq 0$, and suppose that for each subset X of I of cardinality at most n, we have a valuation basis B_X of $\mathcal{U}(K_X((G)) \cap L)$. Suppose that*

$$\bigcup(B_X : X \subseteq I, |X| \leq n)$$

is valuation independent. Then, for each subset X of I of cardinality $n+1$, we may define a valuation basis B_X of $\mathcal{U}(K_X((G)) \cap L)$ such that

$$\bigcup(B_X : X \subseteq I, |X| \leq n+1)$$

is valuation independent.

Proof. As above, using Theorem 4.6 instead of 4.7. □

We can now prove Theorem 2.1 easily.

Proof. By Theorem 4.9, we may take a valuation basis B_X of each valued K-vector space $\mathcal{I}(K_X((G)) \cap L)$ such that whenever $X' \subseteq X$, then $B_{X'} \subseteq B_X$. It follows that the colimit of the B_X, over all finite subsets X of I, is a valuation basis for $\mathcal{I}(L)$. $\square$

The proof of Theorem 2.2 is exactly analogous.

5 Applications

Now, suppose that F is real closed. Applying Theorems 2.1, 2.2, and 3.8 we immediately obtain:

Corollary 5.1. *Assume that F is a real closed field, and G a countable divisible ordered abelian group. There exist $\mathbb{Q}$-valuation bases of $(\mathcal{I}(F(G)^{\sim}), +)$ and $(\mathcal{U}(F(G)^{\sim}), \times)$ with respect to the minimal support valuation $v_{\min}$.*

If F is archimedean, then the $v_{\min}$ valuation coincides with the natural valuation on $F((G))$; we obtain

Corollary 5.2. *Let F be an archimedean real closed field, and G a countable divisible ordered abelian group. Then $(\mathcal{I}(K(G)^{\sim}), +)$ and $(\mathcal{U}(K(G)^{\sim}), \times)$ admit $\mathbb{Q}$-valuation bases with respect to the natural valuation.*

We can now obtain a partial answer to the original question posed in the introduction. Define the *skeleton* of V to be the ordered system of K-vector spaces $S(V) := [\Gamma, \{B(\gamma)\}_{\gamma\in\Gamma}]$, where the *component* $B(\gamma)$ is the K-vector space

$$B(\gamma) = \{x \in V : v(x) \geq \gamma\}/\{x \in V : v(x) > \gamma\} .$$

Now, given an ordered system of K-vector spaces $[\Gamma, \{B(\gamma)\}_{\gamma\in\Gamma}]$, the product $\prod_{\gamma\in\Gamma} B(\gamma)$ is a valued K-vector space, where support(s) and $v_{\min}(s)$ are defined as for fields of power series. The Hahn sum $\coprod_{\gamma\in\Gamma} B(\gamma)$ is the subspace of elements with finite support; its skeleton is precisely the given system $[\Gamma, \{B(\gamma)\}_{\gamma\in\Gamma}]$. By considering "leading coefficients", one sees that if V has skeleton $[\Gamma, \{B(\gamma)\}_{\gamma\in\Gamma}]$ and admits a valuation basis, then $V \simeq \coprod_{\gamma\in\Gamma} B(\gamma)$.

Corollary 5.3. *Let F be an archimedean real closed field, and G a countable divisible ordered abelian group. Then the real closed field $F(G)^{\sim}$ admits a restricted exponential.*

Proof. Since $\mathcal{I}(F(G)^{\sim})$ and $\mathcal{U}(F(G)^{\sim})$ both admit valuation bases, they are both isomorphic as ordered abelian groups to the Hahn sums over their skeleta, which are themselves isomorphic. $\square$

Our final application is to the structure of complements to valuation rings in fields of algebraic series. Observe that for the field $F((G))$, an additive complement to the valuation ring is given by $F((G^{<0}))$, where $F((G^{<0}))$ is the (non-unital) ring of power series with negative support. It follows easily (see [B-K-K]) that for the subfield $L = F(G)^{\sim}$ of $F((G))$, an additive complement to the valuation ring is given by $\mathrm{Neg}(L)$, where $\mathrm{Neg}(K) := F((G^{<0})) \cap L$. We shall call $\mathrm{Neg}(L)$ the *canonical complement* to the valuation ring of L. Note that $F[G^{<0}] \subset \mathrm{Neg}(L)$, where $F[G^{<0}]$ is the *semigroup ring* consisting of power series with negative and finite support. We are interested in understanding when $F[G^{<0}] = \mathrm{Neg}(L)$. In [[B-K-K]; Proposition 2.4], we proved the following

Proposition 5.4. *Assume that G is archimedean and divisible, and that F is a real closed field. Then* $\mathrm{Neg}(L) = F[G^{<0}]$.

On the other hand, in [[B-K-K]; Remark 2.5], we observed that if G is not archimedean, then $F[G^{<0}] \neq \mathrm{Neg}(L)$. The results of this paper imply that:

Proposition 5.5. *Let $L = F(G)^{\sim}$, where F is a real closed field and G is a countable divisible ordered abelian group. Then* $\mathrm{Neg}(L) \simeq F[G^{<0}]$.

Proof. We know that $L = \mathrm{Neg}(L) \oplus \mathcal{O}_L$, and this is a lexicographic decomposition. Now the lexicographic sum of valued vector spaces admits a valuation basis if and only if each summand admits a valuation basis (see [KS1]). It follows that $\mathrm{Neg}(L)$ admits a valuation basis. Clearly $F[G^{<0}]$ also admits a valuation basis. Since $\mathrm{Neg}(L)$ and $F[G^{<0}]$ have the same skeleton, it follows that they are isomorphic as valued vector spaces, and in particular, as ordered groups under addition. $\square$

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A Appendix

We prove our versions of our main theorems, weakened under the assumption that the residue field of our power series field has transcendence degree at most $\aleph_1$. That is, we take F to be an algebraically or real closed field and assume that $\operatorname{trdeg} F \leq \aleph_1$; as in the body of the paper, G denotes a countable ordered abelian group.

In the body of the paper, we write F as the colimit of countable subfields of finite transcendence degree over K; the new assumption $\operatorname{trdeg} F \leq \aleph_1$ enables us to additionally assume this is a linear colimit over countable fields. The linearity renders the prior combinatorial arguments (and supporting technical results) unnecessary, as now we need only verify the optimal approximation property for valued vector space extensions of the form (in the additive case):

$$\mathcal{I}(\, K_\lambda((G)) \cap L\,) \subseteq \mathcal{I}(\, K_{\lambda+1}((G)) \cap L\,)\,.$$

In particular, we may fix a transcendence basis $\{\alpha_\lambda\}_{\lambda<\aleph_1}$ of F over K. Notice that the $\lambda < \aleph_1$ form a directed set. For each $\lambda \leq \aleph_1$, define the subset

$$X_\lambda = \{\alpha_\gamma : \gamma < \lambda\}$$

and the corresponding subfield

$$K_\lambda = K(X_\lambda)^\sim \subset F\,.$$

where $\cdot^\sim$ denotes relative algebraic closure in F. Observe that we have the following colimits of countable objects:

$$\varinjlim \lambda = \aleph_1 \quad \varinjlim K_\lambda = F\,.$$

Moreover, given an intermediate field $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K, the first axiom implies

$$\varinjlim L_\lambda = L \quad \varinjlim \mathcal{I}(\, L_\lambda\,) = \mathcal{I}(\, L\,) \quad \varinjlim \mathcal{U}(\, L_\lambda\,) = \mathcal{U}(\, L\,)\,.$$

where $L_\lambda = K_\lambda((G)) \cap L$.

Theorem A.1 (Bounded Additive)**.** *Let F be an algebraically or real closed field such that $\operatorname{trdeg} F \leq \aleph_1$, K a countably infinite subfield of F and G a countable ordered abelian group. If $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K, then the valued K-vector spaces $(L, +)$ and therefore $(\mathcal{I}(\, L\,), +)$ admit valuation bases.*

Proof. For each λ, define the K-vector space $V_\lambda = (L_\lambda, +)$. We wish to define a valuation basis B_λ for each countable vector space V_λ such that $B_{\lambda'}$ extends B_λ whenever $\lambda \prec \lambda'$.

First, we verify that V_λ has the optimal approximation property in $V_{\lambda+1}$. Indeed, suppose that $f \in V_{\lambda+1} \backslash V_\lambda$; by definition of V_λ, there exists a minimal $q \in \operatorname{support} f$ such the power series coefficient $q(f)$ lies in $K_{\lambda+1} \setminus K_\lambda$. Thus, if h is any approximation to f in V_λ, we necessarily have $v_{\min}(f-h) \leq q$.

Assume for now that F is algebraically closed. By the second TDRP property, we may take an K_λ-rational place P of $K_{\lambda+1}$ such that $\varphi_P(f) \in V_\lambda$; it is then clear that $\varphi_P(f)$ is our desired optimal approximation. On the other hand, if F is real closed, we reduce to the previous case — if f has an optimal approximation g in $V_\lambda \oplus \sqrt{-1}V_\lambda$, then $(g+\sigma(g))/2$ is an optimal approximation to g in V_λ, where σ is the non-trivial element of $\operatorname{Gal}((L \oplus \sqrt{-1}L)/L)$.

Having established the optimal approximation property, we are in a position to define the B_λ via transfinite induction. For $\lambda = 0$, simply select an arbitrary valuation basis B_0 of V_0. For any successor ordinal $\lambda+1$, note that $V_{\lambda+1}$ is countable and thus has countable dimension over V_λ; hence, by Proposition 2.3, the valuation basis B_λ of V_λ extends to one $B_{\lambda+1}$ of $V_{\lambda+1}$. Now for a limit ordinal λ, we see that V_λ is the colimit of the V_ρ for $\rho \prec \lambda$; hence, we may simply define B_λ to be the colimit of the B_ρ for $\rho \prec \lambda$.

Note that the constructed valuation basis for $V_{\aleph_1} = L$ is then simply $B_{\aleph_1}$. ∎

The proof of a multiplicative version is completely analogous — simply define $V_\lambda = (\mathcal{U}(L_\lambda), \times)$ and replace the valuation $v_{\min}$ by $v_{\min}(1-\cdot)$ in the above proof. We thus have

Theorem A.2 (Bounded Multiplicative)**.** *Let F be an algebraically or real closed field of characteristic zero such that* $\operatorname{trdeg} F \leq \aleph_1$*, and G a countable ordered abelian group. If $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over $\mathbb{Q}$ and the group $(\mathcal{U}(L), \times)$ is divisible, then $(\mathcal{U}(L), \times)$ is a valued $\mathbb{Q}$-vector space and admits a $\mathbb{Q}$-valuation basis.*

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